Proc. Natl. Acad. Sci. USA
Vol. 73, No. 7, pp. 2174–2175, July 1976
Mathematics

Maximal functions: Spherical means*

(g-functions/almost-everywhere convergence/wave equation)

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Contributed by Elias M. Stein, March 30, 1976

ABSTRACT Let $\mathcal{M}(f)(x)$ denote the supremum of the averages of f taken over all (surfaces of) spheres centered at x. Then $f \to \mathcal{M}(f)$ is bounded on $L^p(\mathbf{R}^n)$, whenever $p > n/(n-1)$, and $n \geq 3$.

The purpose of this note, and of succeeding ones, is to present some recent results for maximal functions and to discuss several of their applications. This work had as its initial inspiration the idea of Nagel *et al.* (see ref. 1) that the Fourier transform could be used in a decisive way to prove maximal inequalities. In retrospect it is also clear that the spirit of the arguments given below had to some extent been anticipated in Chapter III of ref. 2, where a maximal theorem was proved for symmetric diffusion semi-groups.

For suitable f given on $\mathbf{R}^n$, say $f \in \mathcal{S}$, one may define $M_t(f)(x) = \int_{|y|=1} f(x - ty)\, d\sigma(y)$, which is up to a multiplicative constant the mean-value of f on the sphere of radius t centered at x. ($d\sigma$ is the usual measure on the unit sphere.) We define the corresponding maximal function $\mathcal{M}(f)(x) = \sup_{t>0} |M_t(f)(x)|$.

We are interested in the *a priori* maximal inequality

$$\|\mathcal{M}(f)\|_p \leq A_p \|f\|_p, \; f \in \mathcal{S}. \qquad [1]$$

THEOREM 1. *Suppose $n \geq 3$. Then the inequality* [1] *holds whenever $n/(n-1) < p \leq \infty$. If $p \leq n/(n-1)$ the inequality* [1] *is not valid.*

It is easy to see that when $n = 1$, only the trivial case ($p = \infty$) of [1] holds. When $n = 2$, and $1 \leq p \leq 2$, the inequality also fails. What happens when $n = 2$ (and $2 < p$) remains open.

To see the negative results one merely takes $F(x) = |x|^{-n+1}(\log(1/|x|))^{-1}$, for $0 < |x| \leq 1/2$, and $F(x) = 0$, if $|x| > 1/2$. Then $F \in L^p(\mathbf{R}^n)$, where $p \leq n/(n-1)$, but $\sup_{t>0} |M_t(F)(x)| = \infty$ everywhere. Thus, a simple limiting argument shows that the *a priori* inequality [1] cannot hold for those p.

To prove the positive results and in view of further applications one considers also certain variants of M_t and $\mathcal{M}$. For $\alpha > 0$, let $m_\alpha(x) = (1 - |x|^2)^{\alpha-1}/\Gamma(\alpha)$, when $|x| < 1$ and $m_\alpha(x) = 0$ if $|x| \geq 1$. With $m_{\alpha,t}(x) = m_\alpha(x/t)t^{-n}$, $t > 0$, we define $M_t^\alpha f(x) = (f * m_{\alpha,t})(x)$. Now as is known (see, e.g., ref 3, p. 171) the Fourier transform of m_α is given by

$$\hat{m}_\alpha(\xi) = \pi^{-\alpha+1}|\xi|^{-n/2-\alpha+1} J_{n/2+\alpha-1}(2\pi|\xi|). \qquad [2]$$

Thus, for complex α in general we can also define the operators M_t^α by

$$(M_t^\alpha f)^\wedge(\xi) = \hat{m}_\alpha(\xi t)\hat{f}(\xi), \; f \in \mathcal{S}.$$

We observe that $M_t^0 f$ = a constant multiple of $M_t f$, defined previously. In analogy with $\mathcal{M}$ we define $\mathcal{M}^\alpha$ by $\mathcal{M}^\alpha(f)(x) = \sup_{t>0} |M_t^\alpha(f)(x)|$.

THEOREM 2. *The inequality $\|\mathcal{M}^\alpha(f)\|_p \leq A_{p,\alpha}\|f\|_p$ holds in the following circumstances.*

(a) *if* $1 < p \leq 2$, *when* $\alpha > 1 - n + n/p$.

(b) *if* $2 \leq p \leq \infty$, *when* $\alpha > (1/p)(2 - n)$.

Obviously the postive part of *Theorem 1* is the special case $\alpha = 0$ of *Theorem 2*.

Outline of the proof

The key step in the argument is to consider an appropriate "g-function," whose control is akin to a "Tauberian condition"; this allows one to pass from $\mathcal{M}$ to a standard maximal function.

For each α we fix a function φ which is smooth and has compact support and so that $\hat{\varphi}(0) = \int_{\mathbf{R}^n} \varphi(x)\, dx = \hat{m}_\alpha(0)$. Set $\varphi_t(x) = \varphi(x/t)t^{-n}$. We define $g_\alpha(f)(x)$ by

$$g_\alpha(f)(x) = \left(\int_0^\infty |M_t^\alpha(f)(x) - (f * \varphi_t)(x)|^2 \frac{dt}{t}\right)^{1/2}. \qquad [3]$$

LEMMA 1.

$$\|g_\alpha(f)\|_2 \leq A_\alpha \|f\|_2 \;\; if \;\; \alpha > 1/2 - n/2$$

By Plancherel's formula it suffices to see that

$$\int_0^\infty |\hat{m}_\alpha(\xi t) - \hat{\varphi}(\xi t)|^2 \frac{dt}{t} \leq A_\alpha^2.$$

By homogeneity it is enough to show this when $|\xi| = 1$. The integral is the sum of two parts: one with small t, and the other with large t. The first converges because $\hat{m}_\alpha(0) = \hat{\varphi}(0)$, and both $\hat{m}_\alpha$ and $\hat{\varphi}$ are smooth near the origin. The second part converges because $\hat{m}_\alpha(\xi t) = 0(|t|^{-n/2-\alpha+1/2})$, as $t \to \infty$, in view of Eq. [2]. A direct consequence of the lemma and the fact that $\sup_t |(f * \varphi_t)(x)|$ is majorized by the standard maximal function is the following:

LEMMA 2.

$$\left\|\sup_{t>0}\left\{\frac{1}{t}\int_0^t |M_s^\alpha(f)(x)|^2\, ds\right\}^{1/2}\right\|_2 \leq A_\alpha' \|f\|_2, \quad \alpha > 1/2 - n/2.$$

The next step is to invoke the identity, (whenever $\alpha > \alpha'$),

$$M_t^\alpha(f)(x) = \frac{2}{\Gamma(\alpha - \alpha')}\int_0^1 M_{st}^{\alpha'}(f)(x)(1 - s^2)^{\alpha-\alpha'-1}s^{n+2\alpha'-1}ds \qquad [4]$$

From this and *Lemma 2* it follows that $\|\sup_{t>0}|M_t^\alpha(f)(\,)|\,\|_2 \leq A_\alpha \|f\|_2$, whenever $\alpha > 1 - n/2$, which is *Theorem 2* in the case $p = 2$. The result for $2 \leq p \leq \infty$ then follows from this and the trivial inequality for $p = \infty$, and $Re(\alpha) > 0$ via a convexity argument involving complex α. The case $1 < p \leq 2$ follows similarly, but this time we use the observation that $\mathcal{M}^\alpha(f)(x)$ is essentially majorized by the standard maximal function when $Re(\alpha) \geq 1$. (For similar convexity arguments see, e.g., Chapter VII, §5 of ref. 3.)

* Part I of a series. Part II is the accompanying paper.

Further results

(*i*) We now consider what happens for arbitrary $f \in L^p(\mathbf{R}^n)$ with α satisfying conditions (*a*) or (*b*) of *Theorem 2*. For each fixed t, $t > 0$, the operator $f \rightarrow M_t^\alpha(f)$, initially defined on $\mathcal{S}$, has a unique extension as a bounded operator on L^p. The function $M_t^\alpha(f)(x)$ can now be redefined on a set of x measure zero (for each fixed t) so that the following holds:

PROPOSITION. *With a suitable definition of $M_t^\alpha(f)(x)$ for each t, the function $t \rightarrow M_t^\alpha(f)(x)$ is continuous in $t \in [0,\infty)$, for almost every x. Moreover,* $\|\sup_{t>0} |M_t^\alpha(f)(x)|\|_p \leq A_{p,\alpha}\|f\|_p$.

This proposition is a consequence of *Theorem 2* and a second application of identity [**4**].

(*ii*) We are indebted to A. Cordoba for calling our attention to the connection of *Theorem 1* with solutions of the wave equation. A general result can be stated as follows: Suppose $\alpha = (3/2) - n/2$. If $c_n = (1/2)\pi^{-n/2-1/2}$, then $u(x,t) = c_n t M_t^\alpha(f)(x)$ is a (weak) solution of the wave equation

$$\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = \frac{\partial^2 u}{\partial t^2},$$

with initial conditions $u(x,0) = 0$, $\partial u/\partial t(x,0) = f(x)$. *Theorem 2* combined with (*i*) above then gives us the following analogue of Fatou's theorem for solutions of the wave equation.

COROLLARY. $\lim_{t\rightarrow 0} u(x,t)/t = f(x)$ *almost everywhere, if $f \in L^p(\mathbf{R}^n)$, and $2n/(n+1) < p \leq \infty$. (When $n = 1$, this result holds for $1 \leq p \leq \infty$.) The convergence is also dominatedly in the L^p norm if in addition $p < \infty$, when $n = 1, 2$, or 3, and $p < 2(n-2)/(n-3)$, when $n \geq 4$.*

(*iii*) A. P. Calderón and A. Zygmund have pointed out to the author that a limiting argument applied to *Theorem 1* allows one to show that whenever f is a Borel measurable function in L^p (and $p > n/(n-1)$, with $n \geq 3$), then the integral defining $M_t(f)(x)$ converges for all $t > 0$, except when x belongs to an exceptional set of measure zero. Moreover $\mathcal{M}(f)$ satisfies the inequality [**1**]. As a consequence, whenever E is any set of measure zero in $\mathbf{R}^n$, $n \geq 3$, then for almost every x the intersection of E with any sphere centered at x has $(n-1)$ dimensional measure zero.

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